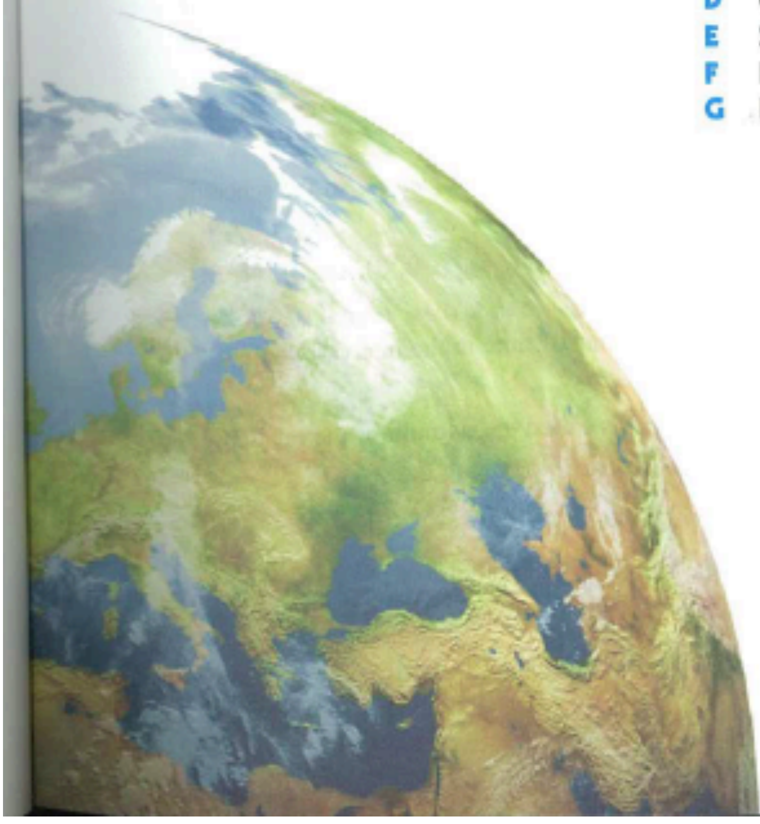
Chapter 2

Functions

Syllabus reference: 2.1, 2.2, 2.5

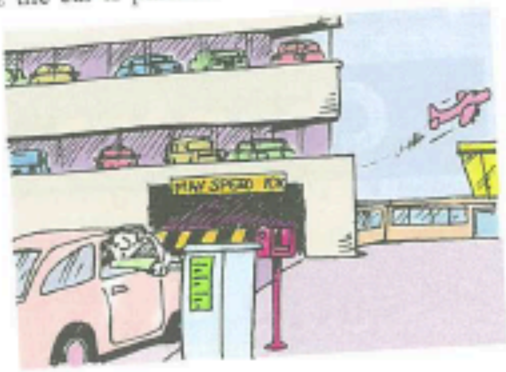
Contents:



A RELATIONS AND FUNCTIONS

The charges for parking a car in a short-term car park at an airport are shown in the table below. The total charge is *dependent* on the length of time t the car is parked.

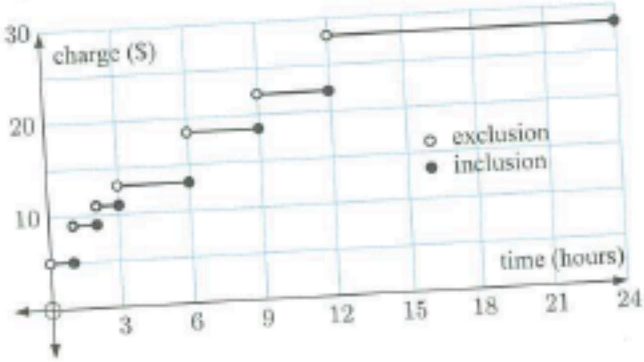
Car park charges	
Time t (hours)	*Charge*
0 - 1 hours	\$5.00
1 - 2 hours	\$9.00
2 - 3 hours	\$11.00
3 - 6 hours	\$13.00
6 - 9 hours	\$18.00
9 - 12 hours	\$22.00
12 - 24 hours	\$28.00



Looking at this table we might ask: How much would be charged for *exactly* one hour? Would it be \$5 or \$9?

To avoid confusion, we could adjust the table or draw a graph. We indicate that 2 - 3 hours really means a time over 2 hours up to and including 3 hours, by writing $2 < t \leqslant 3$ hours.

Car park charges	
Time t (hours)	*Charge*
$0 < t \leqslant 1$ hours	\$5.00
$1 < t \leqslant 2$ hours	\$9.00
$2 < t \leqslant 3$ hours	\$11.00
$3 < t \leqslant 6$ hours	\$13.00
$6 < t \leqslant 9$ hours	\$18.00
$9 < t \leqslant 12$ hours	\$22.00
$12 < t \leqslant 24$ hours	\$28.00



In mathematical terms, we have a relationship between two variables *time* and *charge*, so the schedule of charges is an example of a **relation**.

A relation may consist of a finite number of ordered pairs, such as $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$, or an infinite number of ordered pairs.

The parking charges example is clearly the latter as every real value of time in the interval $0 < t \leqslant 24$ hours is represented.

The set of possible values of the variable on the horizontal axis is called the **domain** of the relation.

For example:
- the domain for the car park relation is $\{t \mid 0 < t \leqslant 24\}$
- the domain of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ is $\{-2, 1, 4\}$.

The set of possible values on the vertical axis is called the **range** of the relation.

For example:
- the range of the car park relation is $\{5, 9, 11, 13, 18, 22, 28\}$
- the range of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ is $\{3, 5, 6\}$.

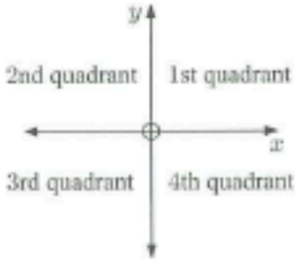
We will now look at relations and functions more formally.

RELATIONS

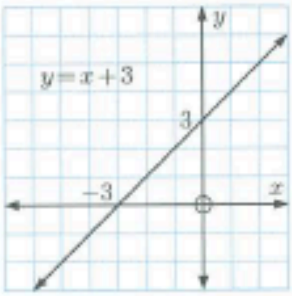
A relation is any set of points which connect two variables.

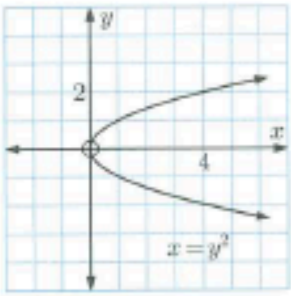
A relation is often expressed in the form of an **equation** connecting the **variables** x and y. In this case the relation is a set of points (x, y) in the **Cartesian plane**.

This plane is separated into four quadrants according to the signs of x and y.



For example, $y = x + 3$ and $x = y^2$ are the equations of two relations. Each equation generates a set of ordered pairs, which we can graph:



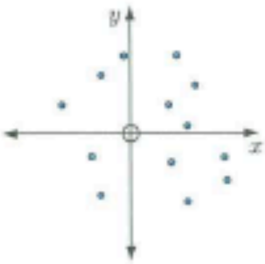


However, a relation may not be able to be defined by an equation. Below are two examples which show this:

(1)

The set of all points in the first quadrant is the relation $x > 0$, $y > 0$

(2)



These 13 points form a relation.

FUNCTIONS

A **function**, sometimes called a **mapping**, is a relation in which no two different ordered pairs have the same x-coordinate or first component.

We can see from the above definition that a function is a special type of relation.

Every function is a relation, but not every relation is a function.

TESTING FOR FUNCTIONS

Algebraic Test:

If a relation is given as an equation, and the substitution of any value for x results in one and only one value of y, then the relation is a function.

For example:

$y = 3x - 1$ is a function, as for any value of x there is only one corresponding value of y

$x = y^2$ is not a function since if $x = 4$ then $y = \pm 2$.

Geometric Test or Vertical Line Test:

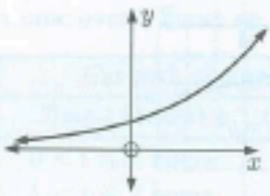
If we draw all possible vertical lines on the graph of a relation, the relation:

- is a function if each line cuts the graph no more than once
- is not a function if at least one line cuts the graph more than once.

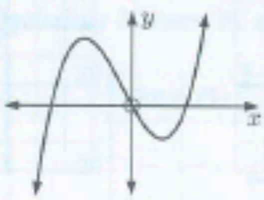
Example 1 Self Tutor

Which of the following relations are functions?

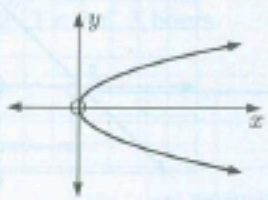
a



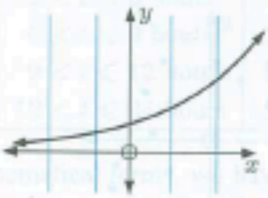
b



c

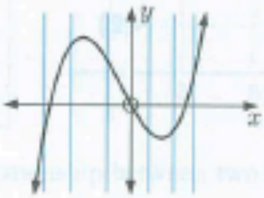


a



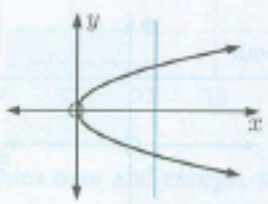
a function

b



a function

c



not a function

DEMO

GRAPHICAL NOTE

- If a graph contains a small **open circle** such as —o—, this point is **not included.**
- If a graph contains a small **filled-in circle** such as ——•, this point **is included.**
- If a graph contains an **arrow head** at an end such as ——→, then the graph continues indefinitely in that general direction, or the shape may repeat as it has done previously.

EXERCISE 2A

1 Which of the following sets of ordered pairs are functions? Give reasons for your answers.

a $\{(1, 3), (2, 4), (3, 5), (4, 6)\}$

b $\{(1, 3), (3, 2), (1, 7), (-1, 4)\}$

c $\{(2, -1), (2, 0), (2, 3), (2, 11)\}$

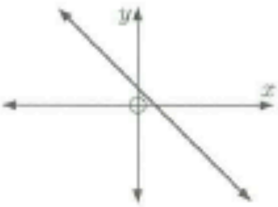
d $\{(7, 6), (5, 6), (3, 6), (-4, 6)\}$

e $\{(0, 0), (1, 0), (3, 0), (5, 0)\}$

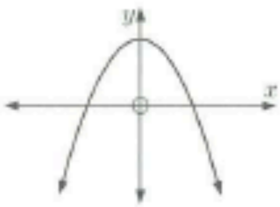
f $\{(0, 0), (0, -2), (0, 2), (0, 4)\}$

2 Use the vertical line test to determine which of the following relations are functions:

a

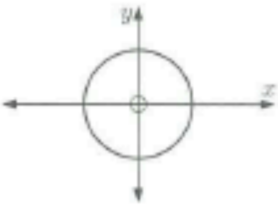


b



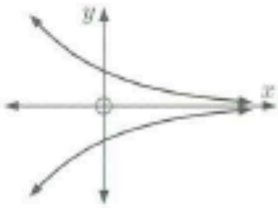
c

d

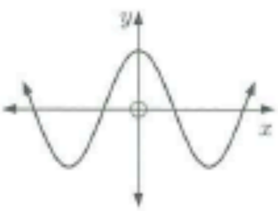


e

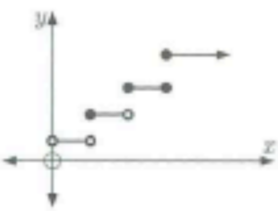
f



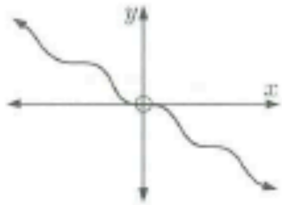
g



h



i

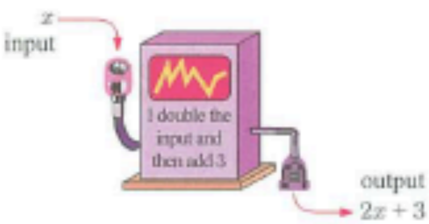


3 Will the graph of a straight line always be a function? Give evidence to support your answer.

4 Give algebraic evidence to show that the relation $x^2 + y^2 = 9$ is not a function.

B FUNCTION NOTATION

Function machines are sometimes used to illustrate how functions behave.



If 4 is the input fed into the machine, the output is $2(4) + 3 = 11$.

The above 'machine' has been programmed to perform a particular function.

If f is used to represent that particular function we can write:

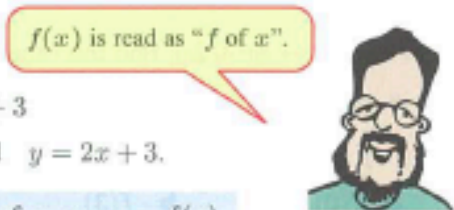
f is the function that will convert x into $2x + 3$.

So, f would convert 2 into $2(2) + 3 = 7$ and
-4 into $2(-4) + 3 = -5$.

This function can be written as:

$$\mathbf{f : x \mapsto \underbrace{2x + 3}}$$

function f such that x is converted into $2x + 3$



Two other equivalent forms we use are $f(x) = 2x + 3$ and $y = 2x + 3$.

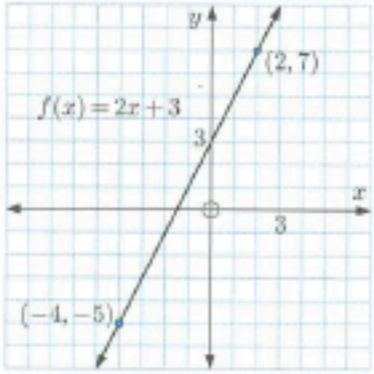
$f(x)$ is the value of y for a given value of x, so $y = f(x)$.

f is the function which converts x into $f(x)$, so we write $f : x \mapsto f(x)$.

$y = f(x)$ is sometimes called the **function value** or **image** of x.

For $f(x) = 2x + 3$:

- $f(2) = 2(2) + 3 = 7$ indicates that the point $(2, 7)$ lies on the graph of the function.
- $f(-4) = 2(-4) + 3 = -5$ indicates that the point $(-4, -5)$ also lies on the graph.



A **linear function** is a function of the form $f(x) = ax + b$ where a, b are real constants. The graph of a linear function is a straight line.

Example 2 Self Tutor

If $f : x \mapsto 2x^2 - 3x$, find the value of: **a** $f(5)$ **b** $f(-4)$

$f(x) = 2x^2 - 3x$

a $f(5) = 2(5)^2 - 3(5)$ {replacing x with (5)}
$= 2 \times 25 - 15$
$= 35$

b $f(-4) = 2(-4)^2 - 3(-4)$ {replacing x with (-4)}
$= 2(16) + 12$
$= 44$

Example 3 Self Tutor

If $f(x) = 5 - x - x^2$, find in simplest form: **a** $f(-x)$ **b** $f(x+2)$

a $f(-x) = 5 - (-x) - (-x)^2$ {replacing x with $(-x)$}
$= 5 + x - x^2$

b $f(x+2) = 5 - (x+2) - (x+2)^2$ {replacing x with $(x+2)$}
$= 5 - x - 2 - [x^2 + 4x + 4]$
$= 3 - x - x^2 - 4x - 4$
$= -x^2 - 5x - 1$

EXERCISE 2B

1 If $f : x \mapsto 3x + 2$, find the value of:

a $f(0)$ **b** $f(2)$ **c** $f(-1)$ **d** $f(-5)$ **e** $f(-\frac{1}{3})$

2 If $f : x \mapsto 3x - x^2 + 2$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(-7)$ **e** $f(\frac{3}{2})$

3 If $g : x \mapsto x - \frac{4}{x}$, find the value of:

a $g(1)$ **b** $g(4)$ **c** $g(-1)$ **d** $g(-4)$ **e** $g(-\frac{1}{2})$

4 If $f(x) = 7 - 3x$, find in simplest form:

a $f(a)$ **b** $f(-a)$ **c** $f(a+3)$ **d** $f(b-1)$ **e** $f(x+2)$ **f** $f(x+h)$

5 If $F(x) = 2x^2 + 3x - 1$, find in simplest form:

a $F(x+4)$ **b** $F(2-x)$ **c** $F(-x)$ **d** $F(x^2)$ **e** $F(x^2-1)$ **f** $F(x+h)$

6 Suppose $G(x) = \frac{2x+3}{x-4}$.

a Evaluate: **i** $G(2)$ **ii** $G(0)$ **iii** $G(-\frac{1}{2})$

b Find a value of x such that $G(x)$ does not exist.

c Find $G(x+2)$ in simplest form.

d Find x if $G(x) = -3$.

7 f represents a function. What is the difference in meaning between f and $f(x)$?

8 The value of a photocopier t years after purchase is given by $V(t) = 9650 - 860t$ euros.

a Find $V(4)$ and state what $V(4)$ means.

b Find t when $V(t) = 5780$ and explain what this represents.

c Find the original purchase price of the photocopier.

9 On the same set of axes draw the graphs of three different functions $f(x)$ such that $f(2) = 1$ and $f(5) = 3$.

10 Find a linear function $f(x) = ax + b$ for which $f(2) = 1$ and $f(-3) = 11$.

11 Given $f(x) = ax + \frac{b}{x}$, $f(1) = 1$, and $f(2) = 5$, find constants a and b.

12 Given $T(x) = ax^2 + bx + c$, $T(0) = -4$, $T(1) = -2$, and $T(2) = 6$, find a, b, and c.

C DOMAIN AND RANGE

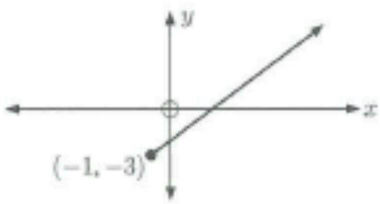
The **domain** of a relation is the set of values of x in the relation.

The **range** of a relation is the set of values of y in the relation.

The domain and range of a relation are often described using **set notation**.

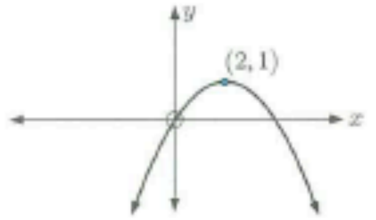
For example:

(1)



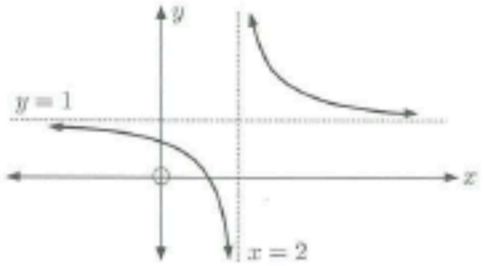
All values of $x \geqslant -1$ are included, so the domain is $\{x \mid x \geqslant -1\}$.
All values of $y \geqslant -3$ are included, so the range is $\{y \mid y \geqslant -3\}$.

(2)



x can take any value,
so the domain is $\{x \mid x \in \mathbb{R}\}$.
y cannot be > 1,
so the range is $\{y \mid y \leqslant 1\}$.

(3)



x can take all values except 2,
so the domain is $\{x \mid x \neq 2\}$.
y can take all values except 1,
so the range is $\{y \mid y \neq 1\}$.

NUMBER LINE GRAPHS

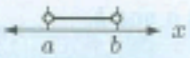
We can illustrate sets of values on a number line graph. For example:

$\{x \mid x \geqslant 3\}$ is read 'the set of all x such that x is greater than or equal to 3' and has number line graph

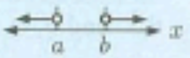
$\{x \mid x < 2\}$ has number line graph

$\{x \mid -2 < x \leqslant 1\}$ has number line graph

$\{x \mid x \leqslant 0 \text{ or } x > 4\}$ has number line graph



For numbers *between* a and b we write $a < x < b$.



For numbers '*outside*' a and b we write $x < a$ or $x > b$.

DOMAIN AND RANGE OF FUNCTIONS

To fully describe a function, we need both a rule *and* a domain.

For example, we can specify $f(x) = x^2$ where $x \geqslant 0$.

If a domain is not specified, we use the **natural domain**, which is the largest part of $\mathbb{R}$ for which $f(x)$ is defined.

For example, consider the domains in the table opposite:

$f(x)$	*Natural domain*
x^2	$x \in \mathbb{R}$
$\sqrt{x}$	$x \geqslant 0$
$\dfrac{1}{x}$	$x \neq 0$
$\dfrac{1}{\sqrt{x}}$	$x > 0$

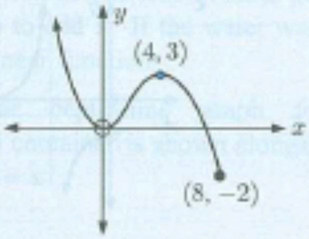
Click on the icon to obtain software for finding the domain and range of different functions.

DOMAIN AND RANGE

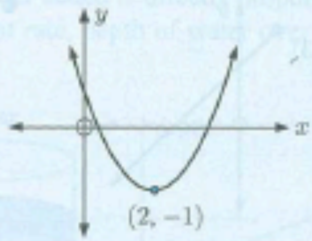
Example 4 **Self Tutor**

For each of the following graphs state the domain and range:

a



b



a Domain is $\{x \mid x \leqslant 8\}$
Range is $\{y \mid y \geqslant -2\}$

b Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \geqslant -1\}$

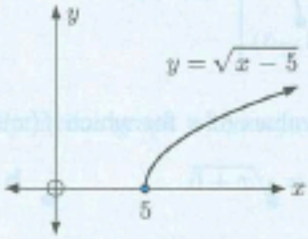
Example 5 **Self Tutor**

State the domain and range of each of the following functions:

a $f(x) = \sqrt{x-5}$ **b** $f(x) = \dfrac{1}{x-5}$ **c** $f(x) = \dfrac{1}{\sqrt{x-5}}$

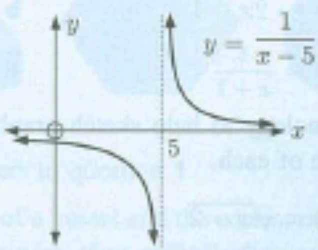
a $\sqrt{x-5}$ is defined when $x - 5 \geqslant 0$
$\therefore\ x \geqslant 5$

$\therefore$ the domain is $\{x \mid x \geqslant 5\}$.
A square root cannot be negative.
$\therefore$ the range is $\{y \mid y \geqslant 0\}$.



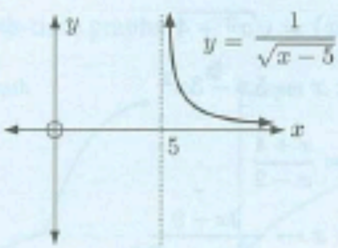
b $\dfrac{1}{x-5}$ is defined when $x - 5 \neq 0$
$\therefore\ x \neq 5$

$\therefore$ the domain is $\{x \mid x \neq 5\}$.
No matter how large or small x is, $y = f(x)$ is never zero.
$\therefore$ the range is $\{y \mid y \neq 0\}$.



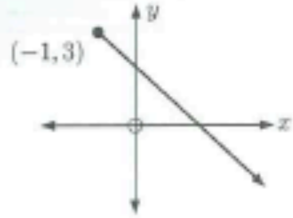
c $\dfrac{1}{\sqrt{x-5}}$ is defined when $x - 5 > 0$
$\therefore\ x > 5$

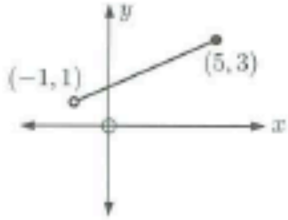
$\therefore$ the domain is $\{x \mid x > 5\}$.
$y = f(x)$ is always positive and never zero.
$\therefore$ the range is $\{y \mid y > 0\}$.

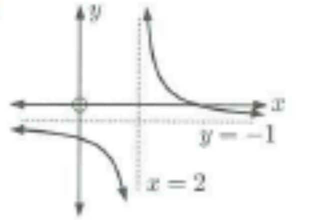


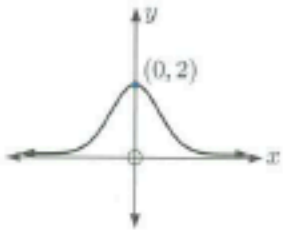
EXERCISE 2C

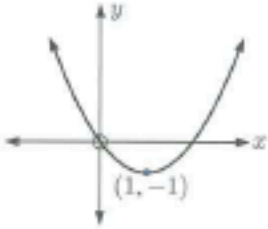
1 For each of the following graphs, find the domain and range:

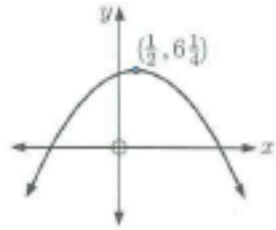
a


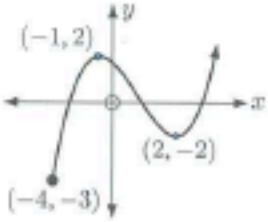
b


c


d


e


f


g


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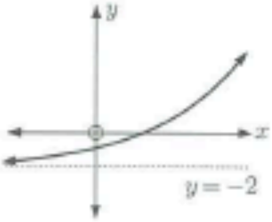

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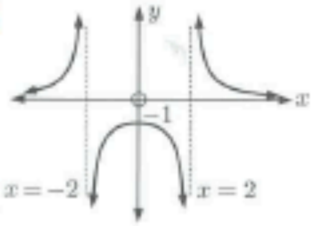


2 State the values of x for which $f(x)$ is defined, and hence state the domain of the function.

a $f(x)=\sqrt{x+6}$ **b** $f: x \mapsto \frac{1}{x^2}$ **c** $f(x)=\frac{-7}{\sqrt{3-2x}}$

3 Find the domain and range of each of the following functions:

a $f: x \mapsto 2x-1$ **b** $f(x)=3$ **c** $f: x \mapsto \sqrt{x}$

d $f(x)=\frac{1}{x+1}$ **e** $f(x)=-\frac{1}{\sqrt{x}}$ **f** $f: x \mapsto \frac{1}{3-x}$

4 Use technology to help sketch graphs of the following functions. Find the domain and range of each.

DOMAIN AND RANGE

a $f(x)=\sqrt{x-2}$ **b** $f: x \mapsto \frac{1}{x^2}$

c $f: x \mapsto \sqrt{4-x}$ **d** $y=x^2-7x+10$

e $f(x)=\sqrt{x^2+4}$ **f** $f(x)=\sqrt{x^2-4}$

g $f: x \mapsto 5x-3x^2$ **h** $f: x \mapsto x+\frac{1}{x}$

i $y=\frac{x+4}{x-2}$ **j** $y=x^3-3x^2-9x+10$

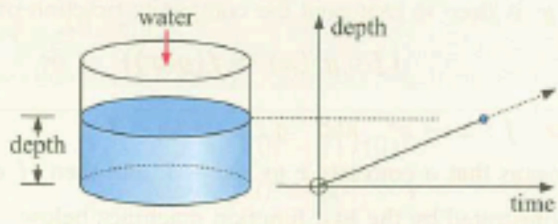
k $f: x \mapsto \frac{3x-9}{x^2-x-2}$ **l** $y=x^2+x^{-2}$

m $y=x^3+\frac{1}{x^3}$ **n** $f: x \mapsto x^4+4x^3-16x+3$

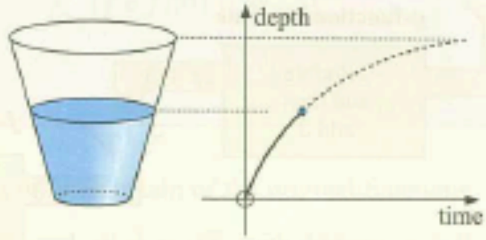
INVESTIGATION 1 — FLUID FILLING FUNCTIONS

When water is added at a **constant rate** to a cylindrical container, the depth of water in the container is a linear function of time. This is because the volume of water added is directly proportional to the time taken to add it. If the water was *not* added at a constant rate, depth of water over time would *not* be a linear function.

The linear depth-time graph for a cylindrical container is shown alongside.



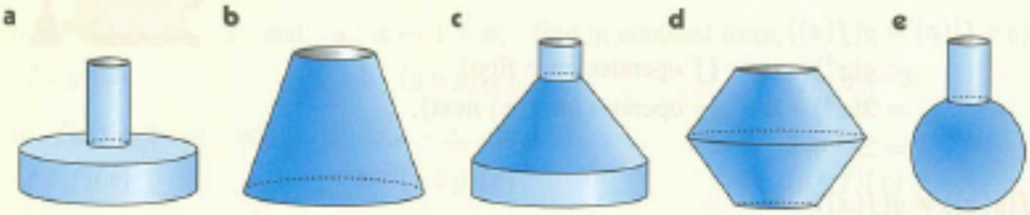
In this investigation we explore the changes in the graph for different shaped containers such as the conical vase.



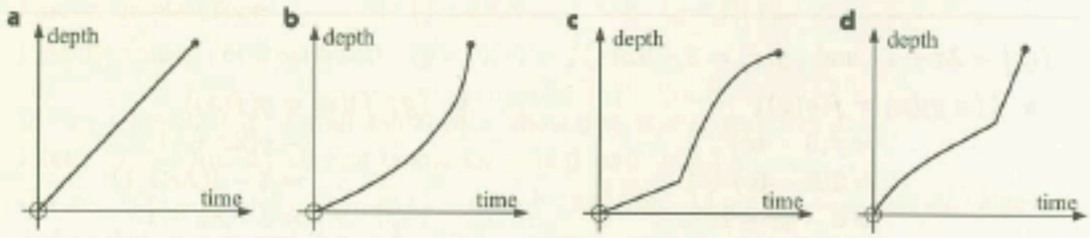
DEMO

What to do:

1. By examining the shape of each container, predict the depth-time graph when water is added at a constant rate.



2. Use the water filling demonstration to check your answers to question **1**.
3. Write a brief report on the connection between the shape of a vessel and the corresponding shape of its depth-time graph. First examine cylindrical containers, then conical, then other shapes. Gradients of curves must be included in your report.
4. Suggest containers which would have the following depth-time graphs:



D COMPOSITE FUNCTIONS

Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the **composite function** of f and g will convert x into $f(g(x))$.

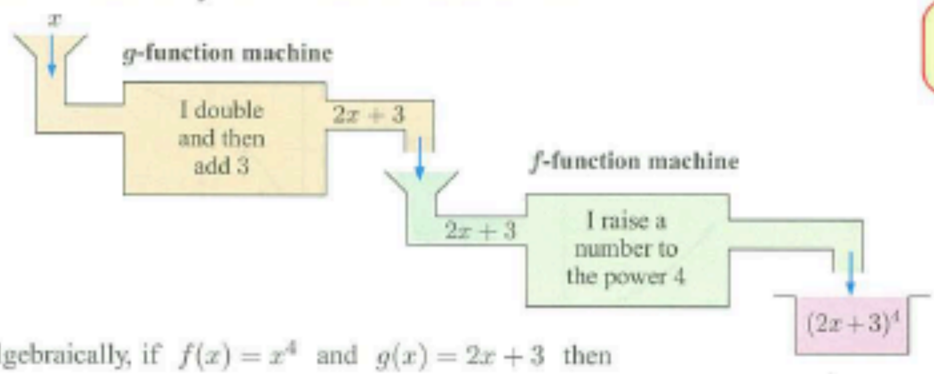
$f \circ g$ is used to represent the composite function of f and g. It means "f following g".

$$(f \circ g)(x) = f(g(x)) \quad \text{or} \quad f \circ g : x \mapsto f(g(x)).$$

Consider $f : x \mapsto x^4$ and $g : x \mapsto 2x + 3$.

$f \circ g$ means that g converts x to $2x + 3$ and then f converts $(2x + 3)$ to $(2x + 3)^4$.

This is illustrated by the two function machines below.



Algebraically, if $f(x) = x^4$ and $g(x) = 2x + 3$ then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) & \\ &= f(2x+3) & \{g \text{ operates on } x \text{ first}\} \\ &= (2x+3)^4 & \{f \text{ operates on } g(x) \text{ next}\}\end{aligned}$$

and
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) & \\ &= g(x^4) & \{f \text{ operates on } x \text{ first}\} \\ &= 2(x^4) + 3 & \{g \text{ operates on } f(x) \text{ next}\} \\ &= 2x^4 + 3 &\end{aligned}$$

So, $f(g(x)) \neq g(f(x))$.

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

Example 6 — Self Tutor

Given $f : x \mapsto 2x + 1$ and $g : x \mapsto 3 - 4x$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

$f(x) = 2x + 1$ and $g(x) = 3 - 4x$

a
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(3 - 4x) \\ &= 2(3 - 4x) + 1 \\ &= 6 - 8x + 1 \\ &= 7 - 8x\end{aligned}$$

b
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x + 1) \\ &= 3 - 4(2x + 1) \\ &= 3 - 8x - 4 \\ &= -8x - 1\end{aligned}$$

In the previous example you should have observed how we can substitute an expression into a function.

If $f(x) = 2x + 1$ then $f(\Delta) = 2(\Delta) + 1$
and so $f(3 - 4x) = 2(3 - 4x) + 1$.

Example 7 **Self Tutor**

Given $f(x) = 6x - 5$ and $g(x) = x^2 + x$, determine:

a $(g \circ f)(-1)$ **b** $(f \circ f)(0)$

a $(g \circ f)(-1) = g(f(-1))$

Now $f(-1) = 6(-1) - 5$
$= -11$

$\therefore$ $(g \circ f)(-1) = g(-11)$
$= (-11)^2 + (-11)$
$= 110$

b $(f \circ f)(0) = f(f(0))$

Now $f(0) = 6(0) - 5$
$= -5$

$\therefore$ $(f \circ f)(0) = f(-5)$
$= 6(-5) - 5$
$= -35$

The domain of the composite of two functions depends on the domain of the original functions.

For example, consider $f(x) = x^2$ with domain $x \in \mathbb{R}$ and $g(x) = \sqrt{x}$ with domain $x \geqslant 0$.

$(f \circ g)(x) = f(g(x))$
$= (\sqrt{x})^2$
$= x$

The domain of $(f \circ g)(x)$ is $x \geqslant 0$, not $\mathbb{R}$, since $(f \circ g)(x)$ is defined using function $g(x)$.

EXERCISE 2D

1 Given $f : x \mapsto 2x + 3$ and $g : x \mapsto 1 - x$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(f \circ g)(-3)$

2 Given $f(x) = 2 + x$ and $g(x) = 3 - x$, find:

a $(f \circ f)(x)$ **b** $(f \circ g)(x)$ **c** $(g \circ f)(x)$

3 Given $f(x) = \sqrt{6 - x}$ and $g(x) = 5x - 7$, find:

a $(g \circ g)(x)$ **b** $(f \circ g)(1)$ **c** $(g \circ f)(6)$

4 Given $f : x \mapsto x^2$ and $g : x \mapsto 2 - x$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.
Find also the domain and range of $f \circ g$ and $g \circ f$.

5 Suppose $f : x \mapsto x^2 + 1$ and $g : x \mapsto 3 - x$.

a Find in simplest form: **i** $(f \circ g)(x)$ **ii** $(g \circ f)(x)$

b Find the value(s) of x such that $(g \circ f)(x) = f(x)$.

6 **a** If $ax + b = cx + d$ for all values of x, show that $a = c$ and $b = d$.
Hint: If it is true for all x, it is true for $x = 0$ and $x = 1$.

b Given $f(x) = 2x + 3$ and $g(x) = ax + b$ and that $(f \circ g)(x) = x$ for all values of x, deduce that $a = \frac{1}{2}$ and $b = -\frac{3}{2}$.

c Is the result in **b** true if $(g \circ f)(x) = x$ for all x?

7 Given $f(x) = \sqrt{1-x}$ and $g(x) = x^2$, find:

a $(f \circ g)(x)$ **b** the domain and range of $(f \circ g)(x)$

E SIGN DIAGRAMS

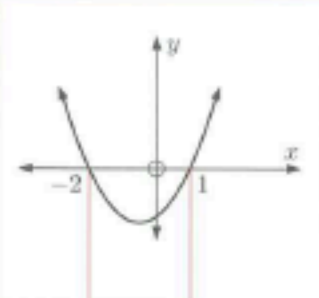
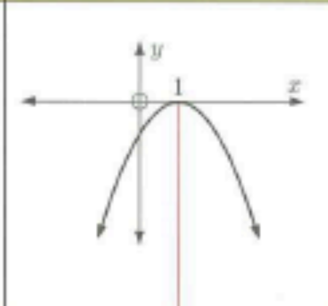
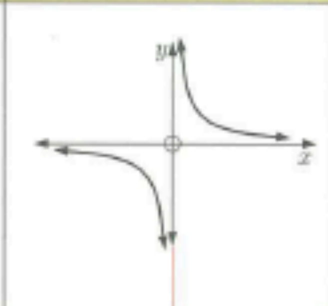
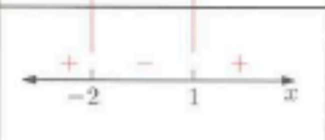
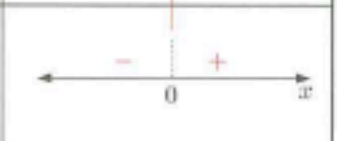
Sometimes we do not wish to draw a time-consuming graph of a function but wish to know when the function is positive, negative, zero or undefined. A **sign diagram** enables us to do this and is relatively easy to construct.

For the function $f(x)$, the sign diagram consists of:

- a **horizontal line** which is really the x-axis
- **positive** $(+)$ and **negative** $(-)$ signs indicating that the graph is **above** and **below** the x-axis respectively
- the **zeros** of the function, which are the x-intercepts of the graph of $y = f(x)$, and the **roots** of the equation $f(x) = 0$
- values of x where the graph is undefined.

DEMO

Consider the three functions given below.

Function	$y = (x+2)(x-1)$	$y = -2(x-1)^2$	$y = \frac{4}{x}$
Graph			
Sign diagram			

From these signs you should notice that:

- A sign change occurs about a zero of the function for single linear factors such as $(x+2)$ and $(x-1)$. This indicates **cutting** of the x-axis.
- No sign change occurs about a zero of the function for squared linear factors such as $(x-1)^2$. This indicates **touching** of the x-axis.
-  indicates that a function is **undefined** at $x = 0$.

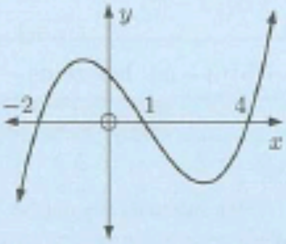
In general:

- when a linear factor has an **odd power** there is a change of sign about that zero
- when a linear factor has an **even power** there is no sign change about that zero.

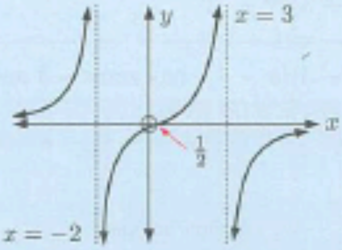
Example 8 **Self Tutor**

Draw sign diagrams for:

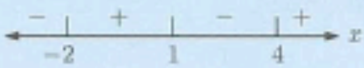
a



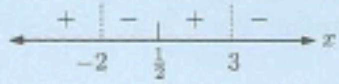
b



a



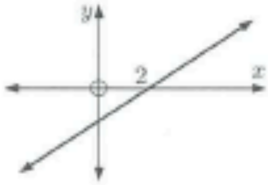
b



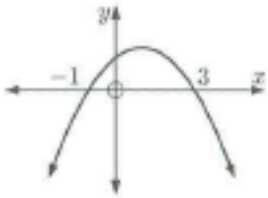
EXERCISE 2E

1 Draw sign diagrams for these graphs:

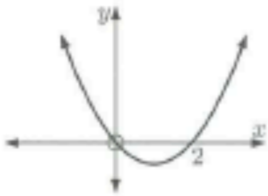
a



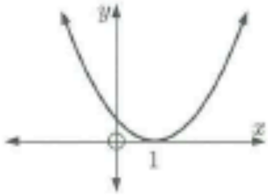
b



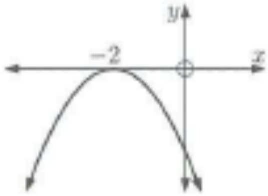
c



d



e



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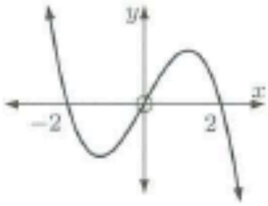
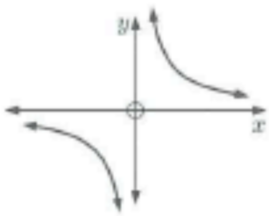


g



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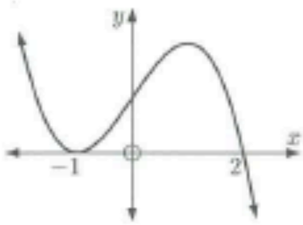
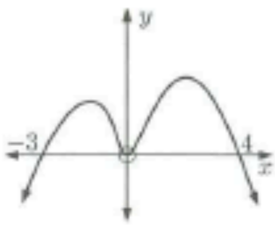


i



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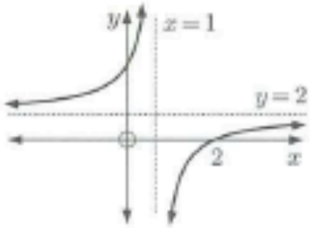

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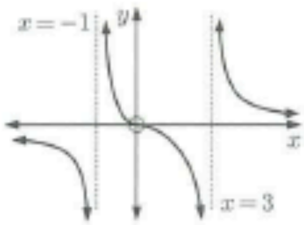

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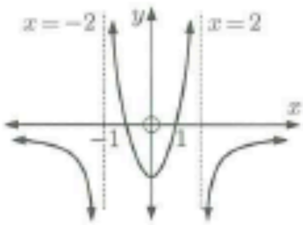


Example 9 **Self Tutor**

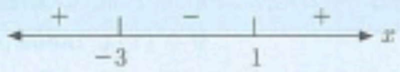
Draw a sign diagram for:

a $(x+3)(x-1)$ **b** $2(2x+5)(3-x)$

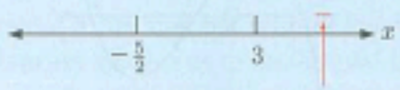
a $(x+3)(x-1)$ has zeros -3 and 1.

We substitute any number > 1.
When $x = 2$ we have $(5)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

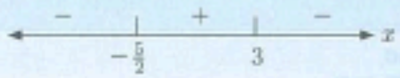


b $2(2x+5)(3-x)$ has zeros $-\frac{5}{2}$ and 3.



We substitute any number > 3.
When $x = 5$ we have $2(15)(-2) < 0$,
so we put a $-$ sign here.

As the factors are single, the signs alternate.



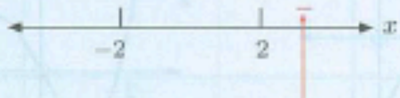
Example 10 **Self Tutor**

Draw a sign diagram for:

a $12-3x^2$ **b** $-4(x-3)^2$

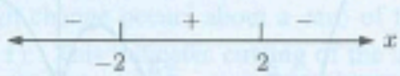
a $12 - 3x^2 = -3(x^2-4)$
$\quad = -3(x+2)(x-2)$

which has zeros -2 and 2.

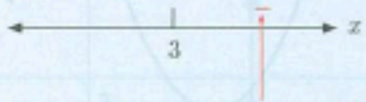


We substitute any number > 2.
When $x = 3$ we have $-3(5)(1) < 0$,
so we put a $-$ sign here.

As the factors are single, the signs alternate.

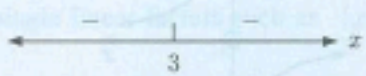


b $-4(x-3)^2$ has zero 3.



We substitute any number > 3.
When $x = 4$ we have $-4(1)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.



2 Draw sign diagrams for:

a $(x+4)(x-2)$ **b** $x(x-3)$ **c** $x(x+2)$

d $-(x+1)(x-3)$ **e** $(2x-1)(3-x)$ **f** $(5-x)(1-2x)$

g x^2-9 **h** $4-x^2$ **i** $5x-x^2$

j x^2-3x+2 **k** $2-8x^2$ **l** $6x^2+x-2$

m $6-16x-6x^2$ **n** $-2x^2+9x+5$ **o** $-15x^2-x+2$

3 Draw sign diagrams for:

a $(x+2)^2$ **b** $(x-3)^2$ **c** $-(x+2)^2$

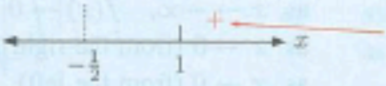
d $-(x-4)^2$ **e** x^2-2x+1 **f** $-x^2+4x-4$

g $4x^2-4x+1$ **h** $-x^2-6x-9$ **i** $-4x^2+12x-9$

Example 11 **Self Tutor**

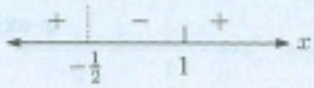
Draw a sign diagram for $\dfrac{x-1}{2x+1}$.

$\dfrac{x-1}{2x+1}$ is zero when $x=1$ and undefined when $x=-\frac{1}{2}$.



When $x=10$, $\dfrac{x-1}{2x+1}=\frac{9}{21}>0$

Since $(x-1)$ and $(2x+1)$ are single factors, the signs alternate.



4 Draw sign diagrams for:

a $\dfrac{x+2}{x-1}$ **b** $\dfrac{x}{x+3}$ **c** $\dfrac{2x+3}{4-x}$ **d** $\dfrac{4x-1}{2-x}$

e $\dfrac{3x}{x-2}$ **f** $\dfrac{-8x}{3-x}$ **g** $\dfrac{(x-1)^2}{x}$ **h** $\dfrac{4x}{(x+1)^2}$

i $\dfrac{(x+2)(x-1)}{3-x}$ **j** $\dfrac{x(x-1)}{2-x}$ **k** $\dfrac{x^2-4}{-x}$ **l** $\dfrac{3-x}{2x^2-x-6}$

F RATIONAL FUNCTIONS

We have seen that a linear function has the form $y=ax+b$.

When we divide a linear function by another linear function, the result is a **rational function**.

Rational functions are characterised by **asymptotes**, which are lines the function gets closer and closer to but never reaches.

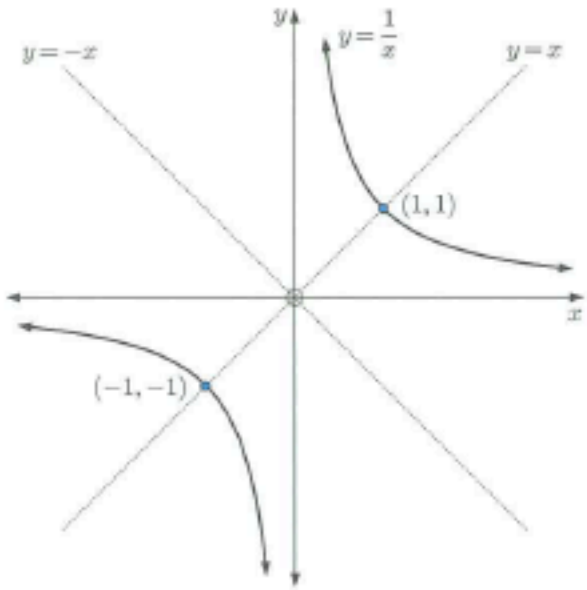
The rational functions we consider in this course can be written in the form $y=\dfrac{ax+b}{cx+d}$. These functions have asymptotes which are horizontal and vertical.

RECIPROCAL FUNCTIONS

A **reciprocal function** is a function of the form $x \mapsto \dfrac{k}{x}$ or $f(x)=\dfrac{k}{x}$, where $k \neq 0$.

The simplest example of a reciprocal function is $f(x)=\dfrac{1}{x}$.

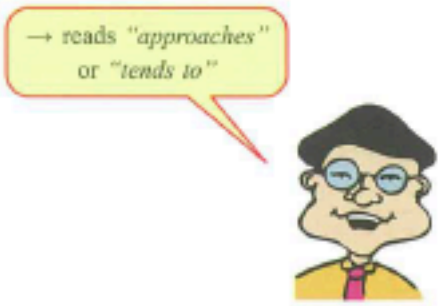
The graph of $f(x) = \frac{1}{x}$ is called a **rectangular hyperbola**.



Notice that:

- $f(x) = \frac{1}{x}$ is undefined when $x = 0$
- The graph of $f(x) = \frac{1}{x}$ exists in the first and third quadrants only.
- $f(x) = \frac{1}{x}$ is symmetric about $y = x$ and $y = -x$.
- as $x \to \infty$, $f(x) \to 0$ (from above)
  as $x \to -\infty$, $f(x) \to 0$ (from below)
  as $x \to 0$ (from the right), $f(x) \to \infty$
  as $x \to 0$ (from the left), $f(x) \to -\infty$
- The **asymptotes** of $f(x) = \frac{1}{x}$ are the x-axis and the y-axis.

GRAPHING PACKAGE



INVESTIGATION 2 RECIPROCAL FUNCTIONS

In this investigation we explore reciprocal functions of the form $y = \frac{k}{x}$, $k \neq 0$.

DEMO

What to do:

1 Use the slider to vary the value of k for $k > 0$.

a Sketch the graphs of $y = \frac{1}{x}$, $y = \frac{2}{x}$, and $y = \frac{4}{x}$ on the same set of axes.

b Describe the effect of varying k on the graph of $y = \frac{k}{x}$.

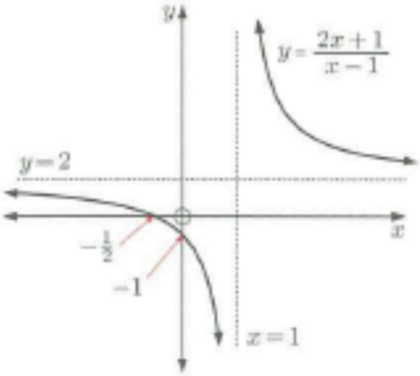
2 Use the slider to vary the value of k for $k < 0$.

a Sketch the graphs of $y = -\frac{1}{x}$, $y = -\frac{2}{x}$, and $y = -\frac{4}{x}$ on the same set of axes.

b Describe the effect of varying k on the graph of $y = \frac{k}{x}$.

RATIONAL FUNCTIONS OF THE FORM $y = \frac{ax+b}{cx+d}$, $c \neq 0$

The graph of $f(x) = \frac{2x+1}{x-1}$ is shown below.



Notice that when $x = 1$, $f(x)$ is undefined.

The graph approaches the vertical line $x = 1$, so $x = 1$ is a vertical asymptote.

Notice that $f(0.999) = -2998$ and $f(1.001) = 3002$.

We can write: as $x \to 1$ (from the left), $f(x) \to -\infty$
as $x \to 1$ (from the right), $f(x) \to \infty$

or as $x \to 1^-$, $f(x) \to -\infty$
as $x \to 1^+$, $f(x) \to \infty$.

To determine the equation of a vertical asymptote, consider the values of x which make the function undefined.

The sign diagram of $y = \frac{2x+1}{x-1}$ is $\overset{+}{\longleftarrow}\big|_{-\frac{1}{2}}\overset{-}{\longrightarrow}\big|_{1}\overset{+}{\longrightarrow} x$ and can be used to discuss the function near its vertical asymptote without having to graph the function.

The graph also approaches the horizontal line $y = 2$, so $y = 2$ is a horizontal asymptote.

Notice that $f(1000) = \frac{2001}{999} \approx 2.003$ and $f(-1000) = \frac{-1999}{-1001} \approx 1.997$

We can write:

as $x \to \infty$, $y \to 2$ (from above) *or* as $x \to \infty$, $y \to 2^+$
as $x \to -\infty$, $y \to 2$ (from below) *or* as $x \to -\infty$, $y \to 2^-$.

We can also write: as $|x| \to \infty$, $f(x) \to 2$.

This indicates that as x becomes very large (either positive or negative) the function approaches the value 2.

To determine the equation of a horizontal asymptote, we consider the behaviour of the function as $|x| \to \infty$.

INVESTIGATION 3 FINDING ASYMPTOTES

What to do:

GRAPHING PACKAGE

1 Use the **graphing package** supplied or a **graphics calculator** to examine the following functions for asymptotes:

a $y = -1 + \frac{3}{x-2}$ **b** $y = \frac{3x+1}{x+2}$ **c** $y = \frac{2x-9}{3-x}$

2 State the domain of each of the functions in **1**.

3 How can we tell directly from the function, what its vertical asymptote is?

DISCUSSION

Can a function cross a vertical asymptote?

Example 12

Self Tutor

Consider the function $y = \frac{6}{x-2} + 4$.

a Find the asymptotes of the function.

b Find the axes intercepts.

c Use technology to help sketch the function, including the features from **a** and **b**.

a The vertical asymptote is $x = 2$.
The horizontal asymptote is $y = 4$.

b When $y = 0$, $\frac{6}{x-2} = -4$

$\therefore\ -4(x-2) = 6$

$\therefore\ -4x + 8 = 6$

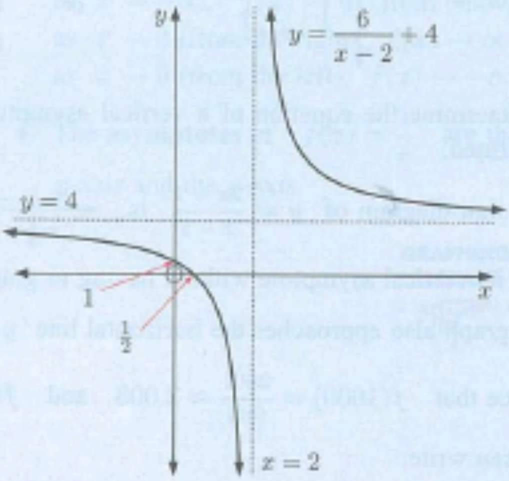
$\therefore\ -4x = -2$

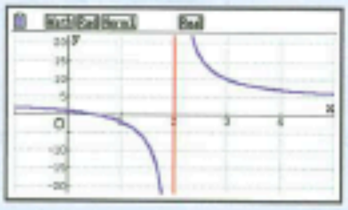
$\therefore\ x = \frac{1}{2}$

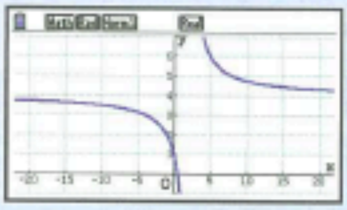
When $x = 0$, $y = \frac{6}{-2} + 4 = 1$

So, the x-intercept is $\frac{1}{2}$ and the y-intercept is 1.

c







Further examples of asymptotic behaviour are seen in exponential, logarithmic, and some trigonometric functions.

EXERCISE 2F

1 For the following functions:

- **i** determine the equations of the asymptotes
- **ii** state the domain and range
- **iii** find the axes intercepts
- **iv** discuss the behaviour of the function as it approaches its asymptotes
- **v** sketch the graph of the function.

a $f : x \mapsto \frac{3}{x-2}$ **b** $y = 2 - \frac{3}{x+1}$ **c** $f : x \mapsto \frac{x+3}{x-2}$ **d** $f(x) = \frac{3x-1}{x+2}$

2 Consider the function $y = \dfrac{ax+b}{cx+d}$, where a, b, c, d are constants and $c \neq 0$.

a State the domain of the function.

b State the equation of the vertical asymptote.

c Show that for $c \neq 0$, $\dfrac{ax+b}{cx+d} = \dfrac{a}{c} + \dfrac{b - \frac{ad}{c}}{cx+d}$.

Hence determine the equation of the horizontal asymptote.

ACTIVITY

Click on the icon to run a card game for rational functions.

GAME

G INVERSE FUNCTIONS

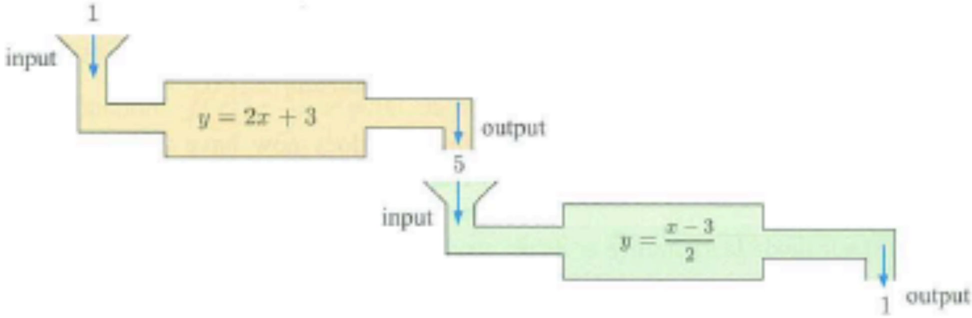
The operations of $+$ and $-$, $\times$ and $\div$, squaring and finding the square root, are **inverse operations** as one undoes what the other does.

For example, $x + 3 - 3 = x$, $x \times 3 \div 3 = x$ and $\sqrt{8^2} = 8$.

The function $y = 2x + 3$ can be "undone" by its *inverse* function $y = \dfrac{x-3}{2}$.

We can think of this as two machines. If the machines are inverses then the second machine *undoes* what the first machine does.

No matter what value of x enters the first machine, it is returned as the output from the second machine.

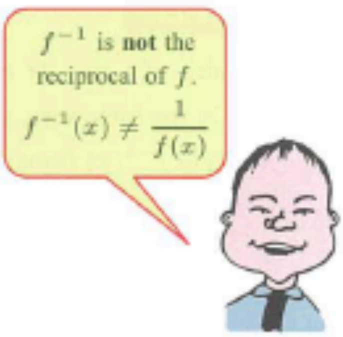


A function $y = f(x)$ *may or may not* have an inverse function.

If $y = f(x)$ has an **inverse function**, this new function:

- is denoted $f^{-1}(x)$
- must indeed be a function, and so must satisfy the vertical line test
- is the reflection of $y = f(x)$ in the line $y = x$
- satisfies $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

The function $y = x$, defined as $f : x \mapsto x$, is the **identity function**.

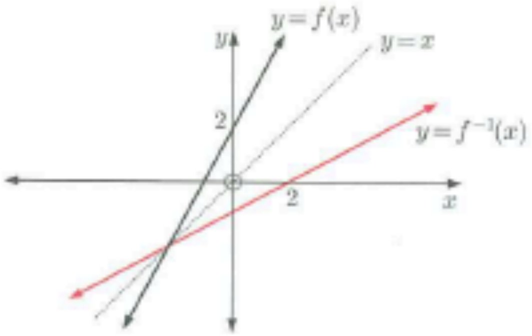


If (x, y) lies on f, then (y, x) lies on f^{-1}. Reflecting the function in the line $y = x$ has the algebraic effect of interchanging x and y.

For example, $f : y = 5x + 2$ becomes $f^{-1} : x = 5y + 2$.

The domain of f^{-1} is equal to the range of f.

The range of f^{-1} is equal to the domain of f.

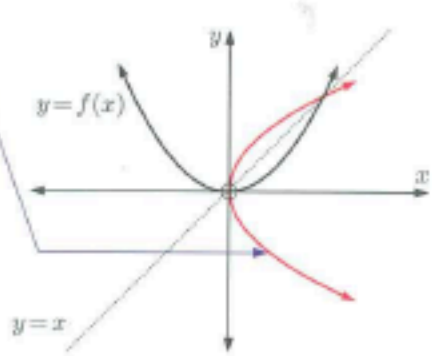


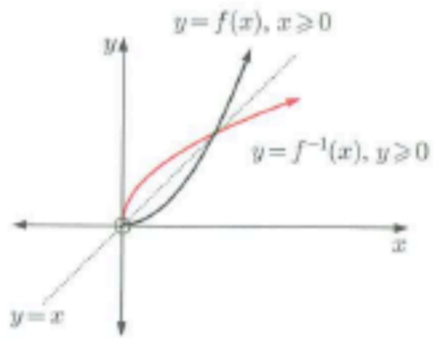
$y = f^{-1}(x)$ is the inverse of $y = f(x)$ as:

- it is also a function
- it is the reflection of $y = f(x)$ in the line $y = x$.

The parabola shown in red is the reflection of $y = f(x)$ in $y = x$, but it is *not* the inverse function of $y = f(x)$ as it fails the vertical line test.

In this case the function $y = f(x)$ does not have an inverse.





Now consider the same function $y = f(x)$ but with the restricted domain $x \geqslant 0$.

The function does now have an inverse function, as shown alongside. However, domain restrictions like this are beyond this course.

Example 13 **Self Tutor**

Consider $f : x \mapsto 2x + 3$.

a On the same axes, graph f and its inverse function f^{-1}.

b Find $f^{-1}(x)$ using:

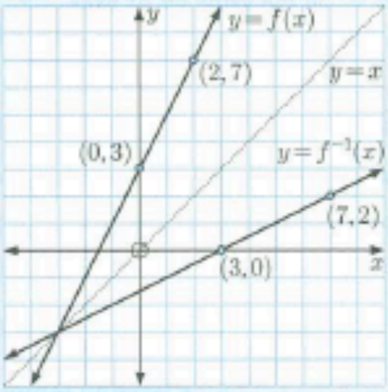
  i coordinate geometry and the gradient of $y = f^{-1}(x)$ from **a**

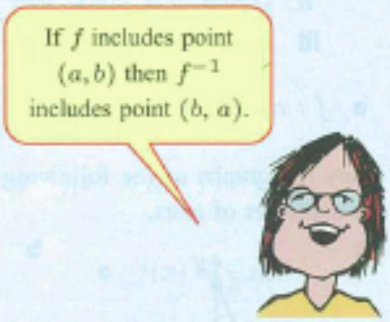
  ii variable interchange.

c Check that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

a $f(x) = 2x + 3$ passes through $(0, 3)$ and $(2, 7)$.

$\therefore$ $f^{-1}(x)$ passes through $(3, 0)$ and $(7, 2)$.





b **i** $y = f^{-1}(x)$ has gradient $\dfrac{2-0}{7-3} = \dfrac{1}{2}$

Its equation is $\dfrac{y-0}{x-3} = \dfrac{1}{2}$

$\therefore\ y = \dfrac{x-3}{2}$

$\therefore\ f^{-1}(x) = \dfrac{x-3}{2}$

ii f is $y = 2x + 3$,

$\therefore\ f^{-1}$ is $x = 2y + 3$

$\therefore\ x - 3 = 2y$

$\therefore\ \dfrac{x-3}{2} = y$

$\therefore\ f^{-1}(x) = \dfrac{x-3}{2}$

c $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$

$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x))\\ &= f\left(\frac{x-3}{2}\right)\\ &= 2\left(\frac{x-3}{2}\right) + 3\\ &= x\end{aligned}$$

$$\begin{aligned}(f^{-1} \circ f)(x) &= f^{-1}(f(x))\\ &= f^{-1}(2x+3)\\ &= \frac{(2x+3)-3}{2}\\ &= \frac{2x}{2}\\ &= x\end{aligned}$$

The reciprocal function $f(x) = \dfrac{1}{x}$, $x \neq 0$, is said to be a **self-inverse function** as $f = f^{-1}$.

This is because the graph of $y = \dfrac{1}{x}$ is symmetrical about the line $y = x$.

Any function which has an inverse, and whose graph is symmetrical about the line $y = x$, is a **self-inverse function**.

Graphics calculator tip:

When graphing f, f^{-1}, and $y = x$ on the same set of axes, it is best to set the scale so that $y = x$ makes a $45°$ angle with both axes.

EXERCISE 2G

1 For each of the following functions f:

- **i** On the same set of axes, graph $y = x$, $y = f(x)$, and $y = f^{-1}(x)$.
- **ii** Find $f^{-1}(x)$ using coordinate geometry and the gradient of $y = f^{-1}(x)$ from **i**.
- **iii** Find $f^{-1}(x)$ using variable interchange.

a $f : x \mapsto 3x + 1$ **b** $f : x \mapsto \dfrac{x+2}{4}$

2 For each of the following functions f:

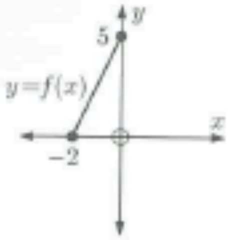
- **i** find $f^{-1}(x)$
- **ii** sketch $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ on the same set of axes
- **iii** show that $(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$, the identity function.

a $f : x \mapsto 2x + 5$ **b** $f : x \mapsto \dfrac{3 - 2x}{4}$ **c** $f : x \mapsto x + 3$

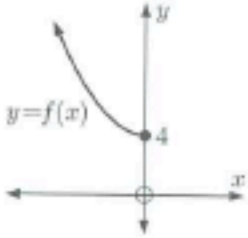
3 Copy the graphs of the following functions and draw the graphs of $y = x$ and $y = f^{-1}(x)$ on the same set of axes.

PRINTABLE GRAPHS

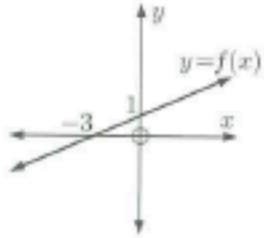
a



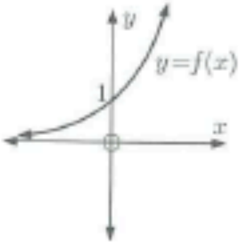
b



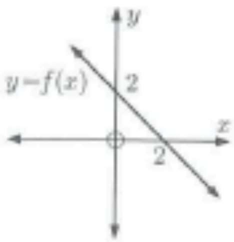
c



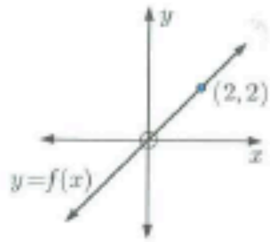
d



e



f



4 For the graph of $y = f(x)$ given in **3 a**, state:

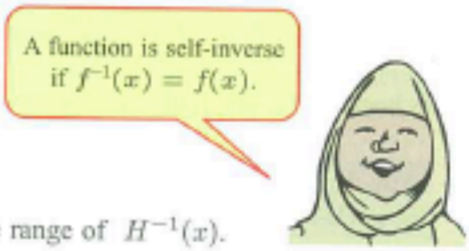
a the domain of $f(x)$ **b** the range of $f(x)$

c the domain of $f^{-1}(x)$ **d** the range of $f^{-1}(x)$.

5 **a** Comment on the results from **3 e** and **f**.

b Draw a linear function that is a self-inverse function.

c Draw a rational function other than $y = \dfrac{1}{x}$, that is a self-inverse function.



6 If the domain of $H(x)$ is $\{x \mid -2 \leqslant x < 3\}$, state the range of $H^{-1}(x)$.

7 Given $f(x) = 2x - 5$, find $(f^{-1})^{-1}(x)$. What do you notice?

8 Sketch the graph of $f : x \mapsto x^3$ and its inverse function $f^{-1}(x)$.

9 Given $f : x \mapsto \dfrac{1}{x}$, $x \neq 0$, find f^{-1} algebraically and show that f is a self-inverse function.

10 Show that $f : x \mapsto \dfrac{3x - 8}{x - 3}$, $x \neq 3$ is a self-inverse function by:

a reference to its graph **b** using algebra.

11 Consider the function $f(x) = \frac{1}{2}x - 1$.

a Find $f^{-1}(x)$. **b** Find: **i** $(f \circ f^{-1})(x)$ **ii** $(f^{-1} \circ f)(x)$.

12 Consider the functions $f : x \mapsto 2x + 5$ and $g : x \mapsto \dfrac{8-x}{2}$.

a Find $g^{-1}(-1)$. **b** Show that $f^{-1}(-3) - g^{-1}(6) = 0$.

c Find x such that $(f \circ g^{-1})(x) = 9$.

13 Consider the functions $f : x \mapsto 5^x$ and $g : x \mapsto \sqrt{x}$.

a Find: **i** $f(2)$ **ii** $g^{-1}(4)$. **b** Solve the equation $(g^{-1} \circ f)(x) = 25$.

14 Given $f : x \mapsto 2x$ and $g : x \mapsto 4x - 3$, show that $(f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$.

15 Which of these functions is a self-inverse function?

a $f(x) = 2x$ **b** $f(x) = x$ **c** $f(x) = -x$

d $f(x) = \dfrac{2}{x}$ **e** $f(x) = -\dfrac{6}{x}$

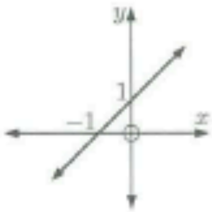
16 The **horizontal line test** says:

For a function to have an inverse function, no horizontal line can cut its graph more than once.

a Explain why this is a valid test for the existence of an inverse function.

b Which of the following functions have an inverse function?

i



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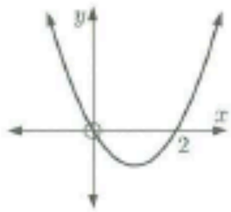
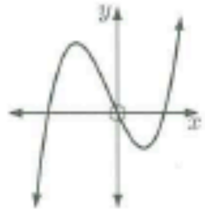


iii

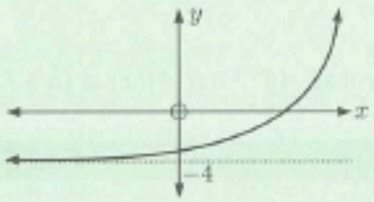


REVIEW SET 2A — NON-CALCULATOR

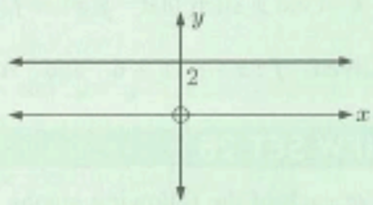
1 For each graph, state:

i the domain **ii** the range **iii** whether the graph shows a function.

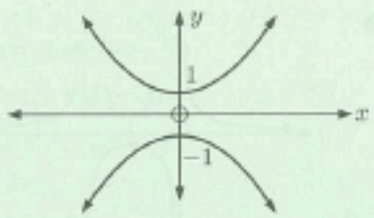
a



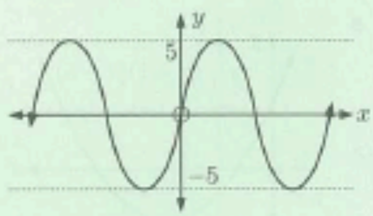
b



c



d



2 If $f(x) = 2x - x^2$, find: **a** $f(2)$ **b** $f(-3)$ **c** $f(-\frac{1}{2})$

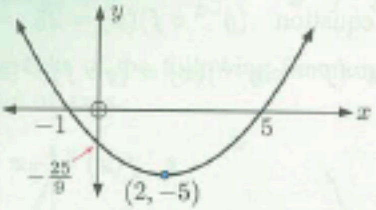
3 Suppose $f(x) = ax + b$ where a and b are constants. If $f(1) = 7$ and $f(3) = -5$, find a and b.

4 If $g(x) = x^2 - 3x$, find in simplest form: **a** $g(x+1)$ **b** $g(x^2-2)$

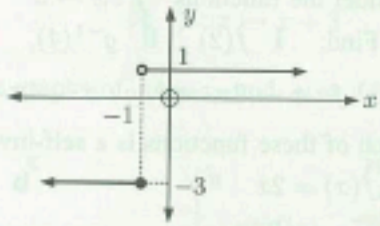
5 For each of the following graphs determine:

i the domain and range **ii** the x and y-intercepts
iii whether it is a function.

a



b



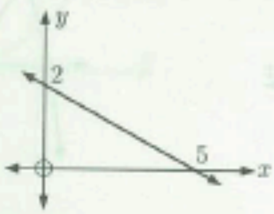
6 Draw a sign diagram for:

a $(3x+2)(4-x)$ **b** $\dfrac{x-3}{x^2+4x+4}$

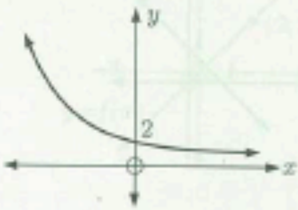
7 If $f(x) = ax + b$, $f(2) = 1$, and $f^{-1}(3) = 4$, find a and b.

8 Copy the following graphs and draw the inverse function on the same set of axes:

a



b



9 Find $f^{-1}(x)$ given that $f(x)$ is: **a** $4x+2$ **b** $\dfrac{3-5x}{4}$

10 Consider $f(x) = x^2$ and $g(x) = 1 - 6x$.

a Show that $f(-3) = g(-\frac{4}{3})$. **b** Find $(f \circ g)(-2)$.

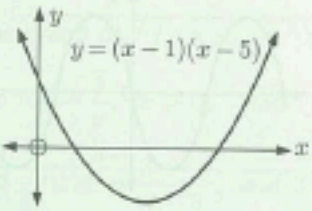
c Find x such that $g(x) = f(5)$.

11 Given $f : x \mapsto 3x + 6$ and $h : x \mapsto \frac{x}{3}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

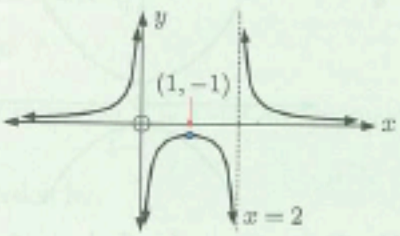
REVIEW SET 2B CALCULATOR

1 For each of the following graphs, find the domain and range:

a



b



2 If $f(x) = 2x - 3$ and $g(x) = x^2 + 2$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

3 Draw a sign diagram for:

a $\dfrac{x^2-6x-16}{x-3}$ **b** $\dfrac{x+9}{x+5}+x$

4 Consider $f(x)=\dfrac{1}{x^2}$.

a For what value of x is $f(x)$ undefined, or not a real number?

b Sketch the graph of this function using technology.

c State the domain and range of the function.

5 Consider the function $f(x)=\dfrac{ax+3}{x-b}$.

a Find a and b given that $y=f(x)$ has asymptotes with equations $x=-1$ and $y=2$.

b Write down the domain and range of $f^{-1}(x)$.

6 Consider the function $f: x \mapsto \dfrac{4x+1}{2-x}$.

a Determine the equations of the asymptotes.

b State the domain and range of the function.

c Discuss the behaviour of the function as it approaches its asymptotes.

d Determine the axes intercepts.

e Sketch the function.

7 Consider the functions $f(x)=3x+1$ and $g(x)=\dfrac{2}{x}$.

a Find $(g\circ f)(x)$.

b Given $(g\circ f)(x)=-4$, solve for x.

c Let $h(x)=(g\circ f)(x)$, $x\neq -\frac{1}{3}$.

i Write down the equations of the asymptotes of $h(x)$.

ii Sketch the graph of $h(x)$ for $-3\leqslant x\leqslant 2$.

iii State the range of $h(x)$ for the domain $-3\leqslant x\leqslant 2$.

8 Consider $f: x\mapsto 2x-7$.

a On the same set of axes graph $y=x$, $y=f(x)$, and $y=f^{-1}(x)$.

b Find $f^{-1}(x)$ using variable interchange.

c Show that $(f\circ f^{-1})(x)=(f^{-1}\circ f)(x)=x$, the identity function.

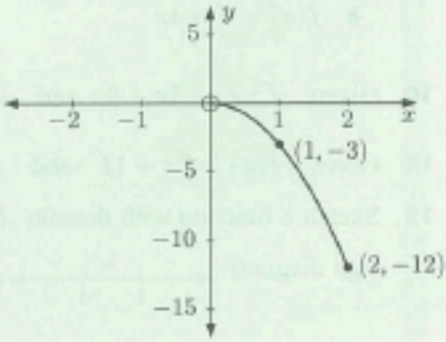
9 The graph of the function $f(x)=-3x^2$, $0\leqslant x\leqslant 2$ is shown alongside.

a Sketch the graph of $y=f^{-1}(x)$.

b State the range of f^{-1}.

c Solve:

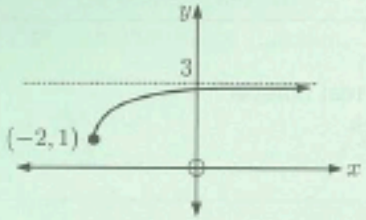
i $f(x)=-10$ **ii** $f^{-1}(x)=1$



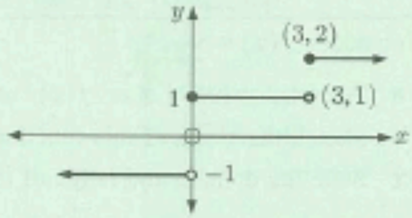
REVIEW SET 2C

1 For each of the following graphs, find the domain and range:

a



b



2 Given $f(x) = x^2 + 3$, find:

a $f(-3)$

b x such that $f(x) = 4$.

3 State the value(s) of x for which $f(x)$ is undefined:

a $f(x) = 10 + \dfrac{3}{2x-1}$

b $f(x) = \sqrt{x+7}$

4 Draw a sign diagram for:

a $f(x) = x(x+4)(3x+1)$

b $f(x) = \dfrac{-11}{(x+1)(x+8)}$

5 Given $h(x) = 7 - 3x$, find:

a $h(2x-1)$ in simplest form

b x such that $h(2x-1) = -2$.

6 If $f(x) = 1 - 2x$ and $g(x) = \sqrt{x}$, find in simplest form:

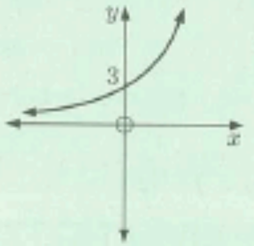
a $(f \circ g)(x)$

b $(g \circ f)(x)$

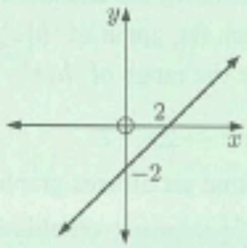
7 Suppose $f(x) = ax^2 + bx + c$. Find a, b, and c if $f(0) = 5$, $f(-2) = 21$, and $f(3) = -4$.

8 Copy the following graphs and draw the graph of each inverse function on the same set of axes:

a



b



9 Find the inverse function $f^{-1}(x)$ for:

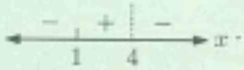
a $f(x) = 7 - 4x$

b $f(x) = \dfrac{3+2x}{5}$

10 Given $f : x \mapsto 5x - 2$ and $h : x \mapsto \dfrac{3x}{4}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

11 Given $f(x) = 2x + 11$ and $g(x) = x^2$, find $(g \circ f^{-1})(3)$.

12 Sketch a function with domain $\{x \mid x \neq 4\}$, range $\{y \mid y \neq -1\}$, and sign diagram



Answer Key

EXERCISE 2A

1 a, d, e **2** a, b, c, e, g, i **3** No, for example $x = 1$

4 No, for example $(0, 3)$ and $(0, -3)$ satisfy the relation.

EXERCISE 2B

1 a 2 b 8 c -1 d -13 e 1

2 a 2 b 2 c -16 d -68 e $\frac{17}{4}$

3 a -3 b 3 c 3 d -3 e $\frac{15}{2}$

4 a $7 - 3a$ b $7 + 3a$ c $-3a - 2$ d $10 - 3b$
  e $1 - 3x$ f $7 - 3x - 3h$

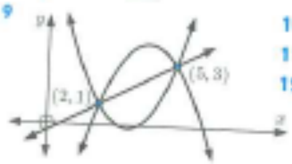
5 a $2x^2 + 19x + 43$ b $2x^2 - 11x + 13$
  c $2x^2 - 3x - 1$ d $2x^4 + 3x^2 - 1$
  e $2x^4 - x^2 - 2$ f $2x^2 + (4h + 3)x + 2h^2 + 3h - 1$

6 a i $-\frac{7}{2}$ ii $-\frac{3}{4}$ iii $-\frac{4}{9}$
  b $x = 4$ c $\dfrac{2x + 7}{x - 2}$ d $x = \frac{9}{5}$

7 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.

8 a 6210 euros, value after 4 years
  b $t = 4.5$ years, the time for the photocopier to reach a value of 5780 euros.
  c 9650 euros

9



10 $f(x) = -2x + 5$

11 $a = 3,\ b = -2$

12 $a = 3,\ b = -1,\ c = -4$

EXERCISE 2C

1 a Domain $= \{x \mid x \geqslant -1\}$, Range $= \{y \mid y \leqslant 3\}$
  b Domain $= \{x \mid -1 < x \leqslant 5\}$, Range $= \{y \mid 1 < y \leqslant 3$
  c Domain $= \{x \mid x \neq 2\}$, Range $= \{y \mid y \neq -1\}$
  d Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid 0 < y \leqslant 2\}$
  e Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \geqslant -1\}$
  f Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \leqslant \frac{25}{4}\}$
  g Domain $= \{x \mid x \geqslant -4\}$, Range $= \{y \mid y \geqslant -3\}$
  h Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y > -2\}$
  i Domain $= \{x \mid x \neq \pm 2\}$,
   Range $= \{y \mid y \leqslant -1$ or $y > 0\}$

2 a $f(x)$ defined for $x \geqslant -6$, Domain $= \{x \mid x \geqslant -6\}$
  b $f(x)$ defined for $x \neq 0$, Domain $= \{x \mid x \neq 0\}$
  c $f(x)$ defined for $x < \frac{3}{2}$, Domain $= \{x \mid x < \frac{3}{2}\}$

3 a Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \in \mathbb{R}\}$
  b Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{3\}$
  c Domain $= \{x \mid x \geqslant 0\}$, Range $= \{y \mid y \geqslant 0\}$
  d Domain $= \{x \mid x \neq -1\}$, Range $= \{y \mid y \neq 0\}$
  e Domain $= \{x \mid x > 0\}$, Range $= \{y \mid y < 0\}$
  f Domain $= \{x \mid x \neq 3\}$, Range $= \{y \mid y \neq 0\}$

4 a Domain $= \{x \mid x \geqslant 2\}$, Range $= \{y \mid y \geqslant 0\}$
  b Domain $= \{x \mid x \neq 0\}$, Range $= \{y \mid y > 0\}$
  c Domain $= \{x \mid x \leqslant 4\}$, Range $= \{y \mid y \geqslant 0\}$
  d Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \geqslant -2\frac{1}{4}\}$
  e Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \geqslant 2\}$
  f Domain $= \{x \mid x \leqslant -2$ or $x \geqslant 2\}$, Range $= \{y \mid y \geqslant 0\}$
  g Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \leqslant \frac{25}{12}\}$
  h Domain $= \{x \mid x \neq 0\}$, Range $= \{y \mid y \leqslant -2$ or $y \geqslant 2\}$
  i Domain $= \{x \mid x \neq 2\}$, Range $= \{y \mid y \neq 1\}$
  j Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \in \mathbb{R}\}$
  k Domain $= \{x \mid x \neq -1$, and $x \neq 2\}$,
   Range $= \{y \mid y \leqslant \frac{1}{3}$ or $y \geqslant 3\}$
  l Domain $= \{x \mid x \neq 0\}$, Range $= \{y \mid y \geqslant 2\}$
  m Domain $= \{x \mid x \neq 0\}$, Range $= \{y \mid y \leqslant -2$ or $y \geqslant 2\}$
  n Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \geqslant -8\}$

EXERCISE 2D

1 a $5 - 2x$ b $-2x - 2$ c 11

2 a $4 + x$ b $5 - x$ c $1 - x$

3 a $25x - 42$ b $\sqrt{8}$ c -7

4 $f(g(x)) = (2 - x)^2$,
  Domain $= \{x \mid x \in \mathbb{R}\}$,
  Range $= \{y \mid y \geqslant 0\}$
  $g(f(x)) = 2 - x^2$,
  Domain $= \{x \mid x \in \mathbb{R}\}$,
  Range $= \{y \mid y \leqslant 2\}$

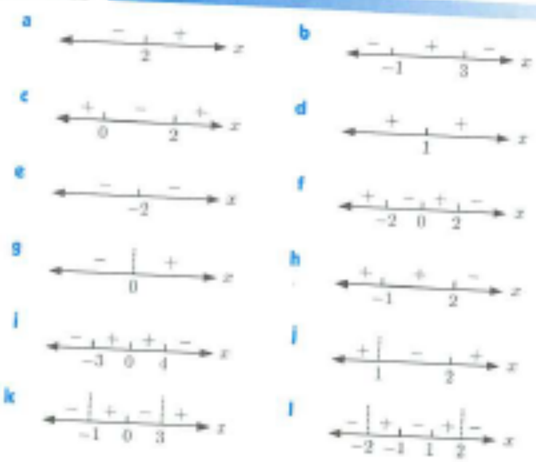
5 a i $x^2 - 6x + 10$ ii $2 - x^2$ b $x = \pm\frac{1}{\sqrt{2}}$

6 a Let $x = 0$, $\therefore$ $b = d$ and so
  $ax + b = cx + b$
  $\therefore$ $ax = cx$ for all x
  Let $x = 1$, $\therefore$ $a = c$
  b $(f \circ g)(x) = [2a]x + [2b + 3] = 1x + 0$ for all x
  $\therefore$ $2a = 1$ and $2b + 3 = 0$
  c Yes, $\{(g \circ f)(x) = [2a]x + [3a + b]\}$

7 a $(f \circ g)(x) = \sqrt{1 - x^2}$
  b Domain $= \{x \mid -1 \leqslant x \leqslant 1\}$, Range $= \{y \mid 0 \leqslant y \leqslant 1\}$

EXERCISE 2E

1



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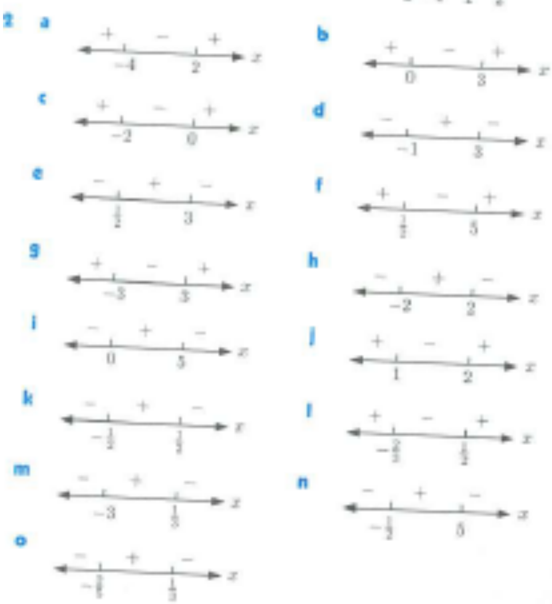
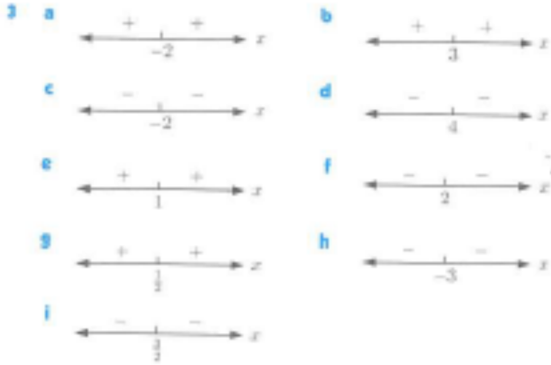
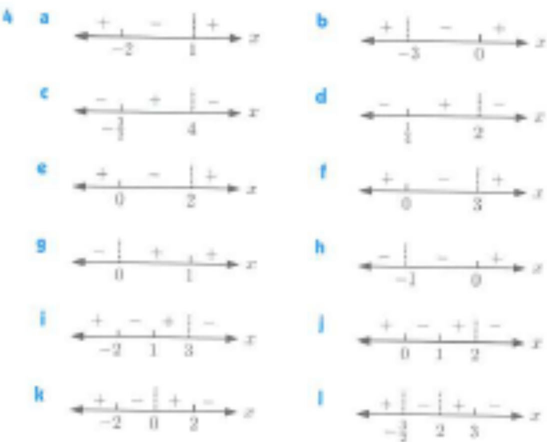
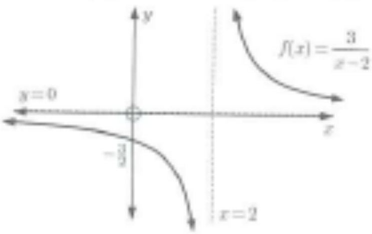
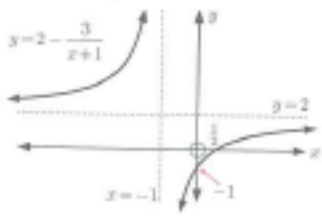


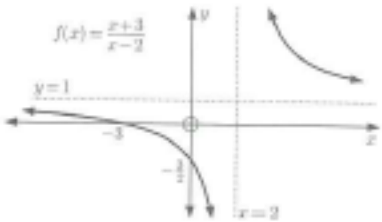
3 

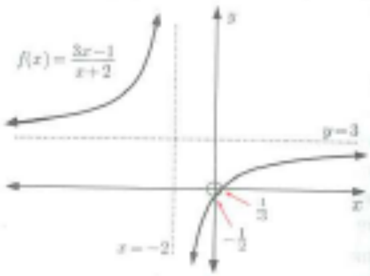
4 

EXERCISE 2F

1 **a** **i** vertical asymptote $x = 2$, horizontal asymptote $y = 0$
ii Domain $= \{x \mid x \neq 2\}$, Range $= \{y \mid y \neq 0\}$
iii no x-intercept, y-intercept $-\frac{3}{2}$
iv as $x \to 2^-$, $y \to -\infty$ as $x \to \infty$, $y \to 0^+$
as $x \to 2^+$, $y \to \infty$ as $x \to -\infty$, $y \to 0^-$
v 

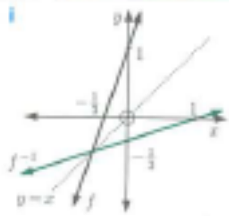
b **i** vertical asymptote $x = -1$, horizontal asymptote $y = 2$
ii Domain $= \{x \mid x \neq -1\}$, Range $= \{y \mid y \neq 2\}$
iii x-intercept $\frac{1}{2}$, y-intercept -1
iv as $x \to -1^-$, $y \to \infty$ as $x \to \infty$, $y \to 2^-$
as $x \to -1^+$, $y \to -\infty$ as $x \to -\infty$, $y \to 2^+$
v 

c **i** vertical asymptote $x = 2$, horizontal asymptote $y = 1$
ii Domain $= \{x \mid x \neq 2\}$, Range $= \{y \mid y \neq 1\}$
iii x-intercept -3, y-intercept $-\frac{3}{2}$
iv as $x \to 2^-$, $y \to -\infty$ as $x \to \infty$, $y \to 1^+$
as $x \to 2^+$, $y \to \infty$ as $x \to -\infty$, $y \to 1^-$
v 

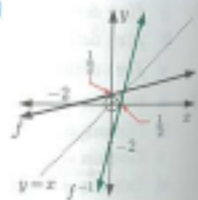
d **i** vertical asymptote $x = -2$, horizontal asymptote $y = 3$
ii Domain $= \{x \mid x \neq -2\}$, Range $= \{y \mid y \neq 3\}$
iii x-intercept $\frac{1}{3}$, y-intercept $-\frac{1}{2}$
iv as $x \to -2^-$, $y \to \infty$ as $x \to \infty$, $y \to 3^-$
as $x \to -2^+$, $y \to -\infty$ as $x \to -\infty$, $y \to 3^+$
v 

2 **a** Domain $= \{x \mid x \neq -\frac{d}{c}\}$ **b** Vertical asymptote $= -\frac{d}{c}$
c Horizontal asymptote $= \frac{a}{c}$

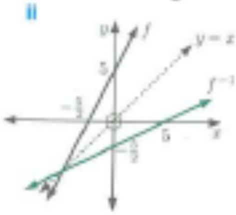
EXERCISE 2G

1 **a** **i** 

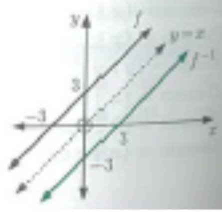
ii, iii $f^{-1}(x) = \frac{x-1}{3}$

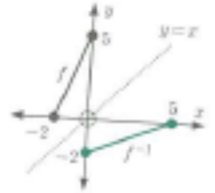
b **i** 

ii, iii $f^{-1}(x) = 4x - 2$

2 **a** **i** $f^{-1}(x) = \frac{x-5}{2}$
ii 

b **i** $f^{-1}(x) = -2x + \frac{3}{2}$
ii 

c **i** $f^{-1}(x)$
$= x - 3$
ii 

3 a


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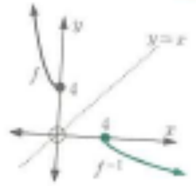

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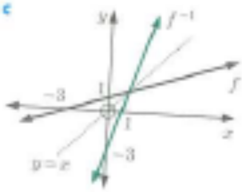

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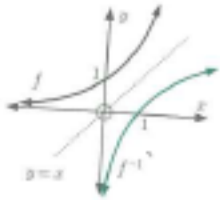
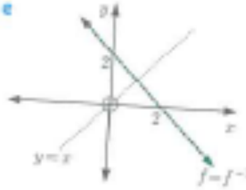


e


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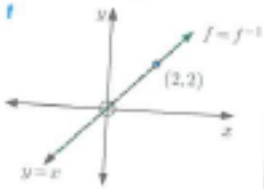
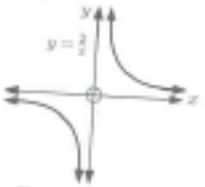


4 a $\{x \mid -2 \leqslant x \leqslant 0\}$ b $\{y \mid 0 \leqslant y \leqslant 5\}$
c $\{x \mid 0 \leqslant x \leqslant 5\}$ d $\{y \mid -2 \leqslant y \leqslant 0\}$

5 a f and f^{-1} are the same. They are self-inverse functions.
b For example, any linear function with slope $= -1$.

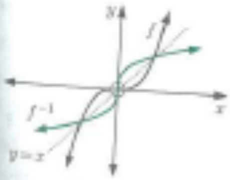

Note: There may be other answers.

c For example, any of $y = \dfrac{a}{x}$, $a \in \mathbb{R}$, $a \neq 0$, 1.


Note: There may be other answers.

6 Range $= \{y \mid -2 \leqslant y < 3\}$

7 $f(x)$ is the same as $(f^{-1})^{-1}(x)$

8


9 $f^{-1}(x) = \dfrac{1}{x}$ and $f(x) = \dfrac{1}{x}$ $\therefore$ $f = f^{-1}$
$\therefore$ f is a self-inverse function

10 a $y = \dfrac{3x-8}{x-3}$ is symmetrical about $y = x$,
$\therefore$ f is a self-inverse function.
b $f^{-1}(x) = \dfrac{3x-8}{x-3}$ and $f(x) = \dfrac{3x-8}{x-3}$
$\therefore$ $f = f^{-1}$ $\therefore$ f is a self-inverse function

11 a $f^{-1}(x) = 2x + 2$
b i $(f \circ f^{-1})(x) = x$ ii $(f^{-1} \circ f)(x) = x$

12 a 10
b $f^{-1}(x) = \dfrac{x-5}{2}$ and $f^{-1}(-3) = -4$
$g^{-1}(x) = 8 - 2x$ and $g^{-1}(6) = -4$
$\therefore$ $g^{-1}(6) - f^{-1}(-3) = 0$
c $x = 3$

13 a i 25 ii 16 b $x = 1$

14 $(f^{-1} \circ g^{-1})(x) = \dfrac{x+3}{8}$ and $(g \circ f)^{-1}(x) = \dfrac{x+3}{8}$

15 a Is not b Is c Is d Is e Is

16 a The inverse function must also be a function and must therefore satisfy the vertical line test, which it can only do if the original function satisfies the horizontal line test.
b i is the only one

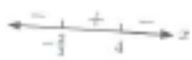
REVIEW SET 2A

1 a i Domain $= \{x \mid x \in \mathbb{R}\}$ ii Range $= \{y \mid y > -4\}$
iii Yes
b i Domain $= \{x \mid x \in \mathbb{R}\}$ ii Range $= \{2\}$ iii Yes
c i Domain $= \{x \mid x \in \mathbb{R}\}$
ii Range $= \{y \mid y \leqslant -1$ or $y \geqslant 1\}$ iii No
d i Domain $= \{x \mid x \in \mathbb{R}\}$
ii Range $= \{y \mid -5 \leqslant y \leqslant 5\}$ iii Yes

2 a 0 b -15 c $-\frac{5}{4}$ 3 $a = -6$, $b = 13$

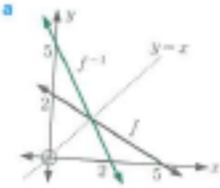
4 a $x^2 - x - 2$ b $x^4 - 7x^2 + 10$

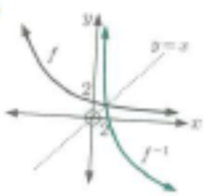
5 a i Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \geqslant -5\}$
ii x-int -1, 5, y-int $-\frac{25}{9}$ iii is a function
b i Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y = 1$ or $-3\}$
ii no x-intercepts, y-intercept 1 iii is a function

6 a


b


7 $a = 1$, $b = -1$

8 a


b


9 a $f^{-1}(x) = \dfrac{x-2}{4}$ b $f^{-1}(x) = \dfrac{3-4x}{5}$

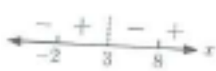
10 a $f(-3) = (-3)^2 = 9$ b 169 c $x = -4$
$g(-\frac{4}{3}) = 1 - 6(-\frac{4}{3}) = 9$

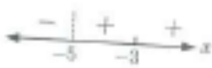
11 $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) = x - 2$

REVIEW SET 2B

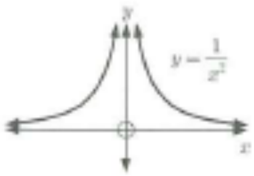
1 a Domain $= \{x \mid x \in \mathbb{R}\}$, Range $= \{y \mid y \geqslant -4\}$
b Domain $= \{x \mid x \neq 0, x \neq 2\}$,
Range $= \{y \mid y \leqslant -1$ or $y > 0\}$

2 a $2x^2 + 1$ b $4x^2 - 12x + 11$

3 a


b


4 **a** $x = 0$ **b**



c Domain $= \{x \mid x \neq 0\}$, Range $= \{y \mid y > 0\}$

5 **a** $a = 2,\ b = -1$

b Domain $= \{x \mid x \neq 2\}$, Range $= \{y \mid y \neq -1\}$

6 **a** vertical asymptote $x = 2$, horizontal asymptote $y = -4$

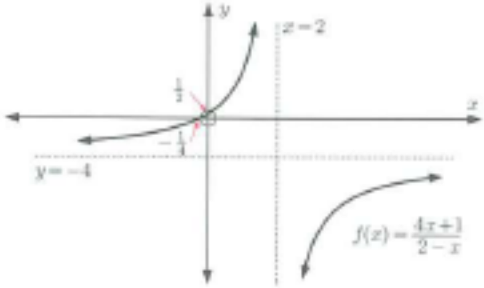
b Domain $= \{x \mid x \neq 2\}$, Range $= \{y \mid y \neq -4\}$

c as $x \to 2^-$, $y \to \infty$ as $x \to \infty$, $y \to -4^-$

as $x \to 2^+$, $y \to -\infty$ as $x \to -\infty$, $y \to -4^+$

d x-intercept $-\frac{1}{4}$, y-intercept $\frac{1}{2}$

e

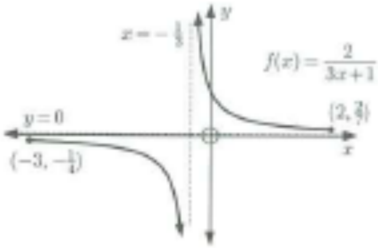


7 **a** $(g \circ f)(x) = \dfrac{2}{3x+1}$ **b** $x = -\frac{1}{2}$

c **i** vertical asymptote $x = -\frac{1}{3}$,

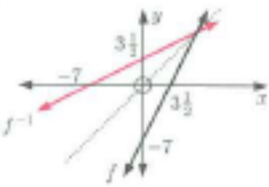
horizontal asymptote $y = 0$

ii



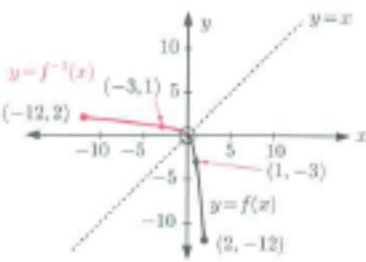
iii Range $= \{y \mid y \leqslant -\frac{1}{4}$ or $y \geqslant \frac{2}{7}\}$

8 **a**



b $f^{-1}(x) = \dfrac{x+7}{2}$

9 **a**



b Range $= \{y \mid 0 \leqslant y \leqslant 2\}$

c **i** $x \approx -1.83$ **ii** $x = -3$

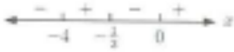
REVIEW SET 2C

1 **a** Domain $= \{x \mid x \geqslant -2\}$, Range $= \{y \mid 1 \leqslant y < 3\}$

b Domain $= \{x \in \mathbb{R}\}$, Range $= \{y \mid y = -1, 1 \text{ or } 2\}$

2 **a** 12 **b** $x = \pm 1$ **3** **a** $x = \frac{1}{2}$ **b** $x < -7$

4 **a**

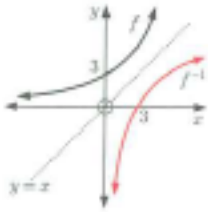


b

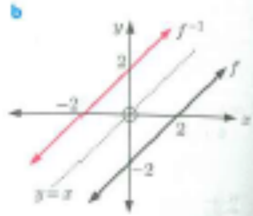
5 **a** $10 - 6x$ **b** $x = 2$ **6** **a** $1 - 2\sqrt{x}$ **b** $\sqrt{1 - 2x}$

7 $a = 1,\ b = -6,\ c = 5$

8 **a**



b



9 **a** $f^{-1}(x) = \dfrac{7-x}{4}$ **b** $f^{-1}(x) = \dfrac{5x-3}{2}$

10 $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) = \dfrac{4x+6}{15}$ **11** 16

12